Diffusion in a stochastic magnetic field.

We consider a stochastic differential equation for a charged particle in a stochastic magnetic field, known as A-Langevin equation. The solution of the equation is found, and the velocity correlation function $\langle u(t)^i \, u(0)^j \rangle$ is calculated in Corrsin approximation. A corresponding diffusion constant is estimated. We observe different transport regimes, such as quasilinear- or Bohm-type diffusion, depending on the parameters of plasma.

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The problem of transport in a magnetic field is in focus of a number of theoretical and experimental works performed during the last several decades. The reason for such extraordinary interest to this problem is caused by the problem of still unsuccessful plasma confinement. A very strong deviation of the diffusion rate from naive classical predictions is due to the nonlinear effects, which have to be taken into account. On the other hand, the character of transport phenomena is often very counterintuitive and paradoxical, what cautions one to be especially careful with any theoretical predictions.

In our work, we use the stochastic differential equations approach to the problem of transport of charged particles in a magnetic field [1]. There are two traditional ways of considering this problem. The first one usually is referred to as V-Langevin equations. In this approach one considers stochastic equations for a guiding center of a test particle in a drift approximation [2–5].

We concentrated on the second approach named A-Langevin equations, that invokes an exact equation of motion of a single test particle, for which interaction forces are mimicked by phenomenological damping and acceleration terms. On the base of the solution of the equation of motion one can calculate a velocity correlation function that leads to the diffusion tensor. Generally the exact solution of the problem is not possible, or at least, extremely complicated. Nevertheless it is still possible to make some estimations in different limiting cases assuming that the perturbation of the magnetic field is weak.

The paper is organized as follows. In the first section we give the mathematical formulation of the problem and present the solution of the equation of motion as well as the velocity correlation function, expressed through the Lagrange correlator of the magnetic field. In the second part we estimate the Lagrange magnetic field correlator in the Corrsin approximation. This allows us to reformulate the problem in terms of a differential equation for the mean square displacement. In the third part we find some particular solutions of this equation and present the corresponding diffusion constants. All mathematical details of the derivations are placed in appendices.

1 Equation of motion and formulation of the problem

The equation of motion of a charged particle with the charge Ze in a magnetic field $\mathbf{B}(t)$, that experiences damping and random acceleration ($-\nu\,\mathbf{u}(t)$ and $\mathbf{a}(t)$ respectively) is given by:

$$\dot{\mathbf{u}}(t) = \frac{Z\,e}{mc}\,[\mathbf{u}(t)\times\mathbf{B}(t)] - \nu\,\mathbf{u}(t) + \mathbf{a}(t)\,. \tag{1.1}$$

where ν is an effective collisions frequency.

This is a stochastic differential equation, known as A-Langevin equation. There are two stochastic functions in this equation: the stochastic magnetic field $\mathbf{B}(t)$ and the random acceleration $\mathbf{a}(t)$. To close the system of equations, the stochastic properties of these functions should be defined. We assume that both are Gaussian processes, which means that the first and the second order correlation functions provide a complete statistical description of those functions. Concerning the magnetic field it means that we consider only the regions of plasma with completely chaotic magnetic field, not containing structures like KAM surfaces or islands.

For the random acceleration $\mathbf{a}(t)$ we chose the white noise approximation, i.e.

$$\langle a(t)^i\rangle = 0\,,\quad \langle a(t_1)^i\,a(t_2)^j\rangle = A\,\delta^{ij}\,\delta(t_1-t_2)\,. \tag{1.2}$$

It is well known from the theory of Brownian particle that the equilibrium thermal velocity v_{th} is related to the collisions frequency ν and value A as $v_{th}^2 = A/2\,\nu\,.$ This relation remains valid for charged particles in a magnetic field as well, since equilibrium thermal velocity is not affected by the Lorenz force.

The magnetic field is supposed to consist of two parts: a constant component B_0 directed along the z-axis and a small perturbation with zero average:

$$\mathbf{B}(\mathbf{r},t) = B_0\,\mathbf{e}_z + \epsilon\,B_0\,\vec{\zeta}(\mathbf{r},t)\,;\qquad \text{where}\quad \langle|\vec{\zeta}|^2\rangle = 1\,,\quad \langle\vec{\zeta}\,\rangle = 0\,.$$

The perturbation parameter ϵ is supposed to be small. The *Eulerian* correlation function (ECF) of the magnetic field

$$\mathcal{E}(|\delta\mathbf{r}|,|\delta t|) = \langle\zeta(\mathbf{r}_1,t_1)^i\,\zeta(\mathbf{r}_2,t_2)^j\rangle\,,\qquad \delta\mathbf{r} = \mathbf{r}_1-\mathbf{r}_2\,,\quad \delta t = t_1-t_2\,, \tag{1.3}$$

will be specified in subsequent sections. As we will see later, the ECF appears in the calculation in implicit form. The *Lagrange* correlation function (LCF)

$$\mathcal{L}^{ij}(|\delta t|) = \langle\zeta(\mathbf{r}_1(t_1),t_1)^i\,\zeta(\mathbf{r}_2(t_2),t_2)^j\rangle \tag{1.4}$$

appears explicitly instead. In contrast to the ECF, which is calculated in fixed points of space, the LCF is taken in points of the stochastic trajectory, and thus cannot be obtained from the stochastic properties of the magnetic field solely. To perform the calculation, we have to make a hypothesis of statistical independence of the LCF from other stochastic processes. Within the framework of this hypothesis one can perform averaging over all stochastic processes independently from one

another. After the averaging, the mean displacement $\langle \mathbf{r}^2(t) \rangle$ as well as the velocity correlation function will be found from the self consistency condition.

The object of our interest is the "running" diffusion coefficient, that is defined in the usual way as

$$D_i(t) = \frac{1}{2} \frac{d}{dt} \langle r_i^2(t) \rangle ; \qquad i = x,\, y,\, z\,.$$

The mean square displacement (MSD) $\langle r_i^2(t) \rangle$ can be derived in turn from the velocity correlation matrix:

$$\left\langle r^i(t)\, r^j(t) \right\rangle = \int_0^t\!\!\int_0^t dt_1\, dt_2\, \left\langle u^i(t_1)\, u^j(t_2) \right\rangle = 2 \int_0^t (t-\tau)\, \left\langle u^i(\tau)\, u^j(0) \right\rangle\, d\tau\,. \tag{1.5}$$

Combining the last two formulas, we write the diffusion tensor as

$$D_{ij}(t) = \int_0^t \left\langle u^i(\tau)\, u^j(0) \right\rangle\, d\tau\,. \tag{1.6}$$

As one sees, the problem is now reduced to the calculation of the velocity correlator $\langle u^i(\tau)\, u^j(0) \rangle$ in the Lagrange frame of reference.

The solution of the Eq. (1.1) can be found straightforwardly. First we transform this equation into standard form, introducing the generators of the $SO(3)$ group $\hat{\mathbf{L}} = (\hat{L}_i)_{jk} = -\epsilon_{ijk}$. The corresponding finite rotation matrices are $R_i(t) = \exp(t\, \hat{L}_i)$. We define the "magnetic field operator" as

$$\hat{H}(t) = \vec{\zeta}(\mathbf{r}(t), t) \cdot \hat{\mathbf{L}} = \zeta_1(\mathbf{r}(t), t)\, \hat{L}_1 + \zeta_2(\mathbf{r}(t), t)\, \hat{L}_2 + \zeta_3(\mathbf{r}(t), t)\, \hat{L}_3\,, \tag{1.7}$$

where $\mathbf{r}(t)$ is the particle trajectory. Introducing the unperturbed part of Lamor frequency $\Omega_0 = Z\, e\, B_0/(mc)$, we can now rewrite Eq. (1.1) as

$$\dot{\mathbf{u}}(t) = [-\nu - \Omega_0\, \hat{L}_3 - \epsilon\, \Omega_0\, \hat{H}(t)]\, \mathbf{u}(t) + \mathbf{a}(t)\,. \tag{1.8}$$

In terms of the Green function the solution is

$$\mathbf{u}(t) = e^{-\nu\, t}\, R_3(-\Omega_0\, t) \left\{ G(t)\, \mathbf{u}_0 + \int_0^t G(t', t)\, \mathbf{a}_{int}(t')\, dt' \right\}, \tag{1.9}$$

where the propagator $G(t_2, t_1)$ is defined as:

$$G(t_2, t_1) = T\, \exp\left(-\epsilon\, \Omega_0 \int_{t_2}^{t_1} \hat{H}_{int}(\tau)\, d\tau \right)\; ; \qquad G(t) := G(0, t)\,, \tag{1.10}$$

and $T\, \exp(\,)$ denotes a time-ordered exponent. The values $\hat{H}_{int}(t)$ and $\mathbf{a}_{int}(t)$ are magnetic field operator and random acceleration in interaction representation:

$$\hat{H}_{int}(t) = R_3(\Omega_0\, t)\, \hat{H}(t)\, R_3(-\Omega_0\, t)\,; \quad \mathbf{a}_{int}(t) = e^{\nu\, t}\, R_3(\Omega_0\, t)\, \mathbf{a}(t)\,. \tag{1.11}$$

Now we can calculate the velocity correlation function. On the base of the solution (1.9) we construct a product $u^i(t_1)\, u^j(t_2)$, that should be averaged over all the stochastic functions – random

acceleration $\mathbf{a}(t)$, initial velocities $\mathbf{u}_0$ and magnetic field fluctuations $\vec{\zeta}(t)$. The expression for the product $u^i(t_1)\, u^j(t_2)$ becomes rather bulky though straightforward, so we relegate the calculation of the velocity correlator to the appendix A[1]. After averaging, the velocity correlation function becomes a reduced matrix with diagonal elements:

$$\langle u(t_1)^i\, u(t_2)^i \rangle = \mathbf{Re}\left[v_{th}^2 e^{-\nu\tau-\gamma(\tau)} e^{-i\,\overline{\Omega}_0\,\tau}\right] + \epsilon^2 v_{th}^2 \mathcal{L}_x(\tau)\, e^{-\nu\tau-\gamma(\tau)}, \qquad i = 1,2 \tag{1.12a}$$

$$\langle u(t_1)^i\, u(t_2)^i \rangle = v_{th}^2\, e^{-\nu\,\tau} + 2\,\epsilon^2\, v_{th}^2\, e^{-\nu\,\tau}\, \mathcal{L}_x(\tau)\, \cos\overline{\Omega}_0\tau\,, \qquad i = 3 \tag{1.12b}$$

where $\tau = |t_1 - t_2|,\ v_{th}$ is the equilibrium thermal velocity, $\overline{\Omega}_0 = \Omega_0 + \epsilon^2\, \Omega_0$ is the renormalized Lamor frequency, $\mathcal{L}_x(t), \mathcal{L}_y(t), \mathcal{L}_z(t)$ are the diagonal elements of the LCF, and

$$\gamma(t) = \epsilon^2\, \Omega_0^2 \int_0^t (t-\tau)\mathcal{L}_z(\tau)\, d\tau = \epsilon^2\, \Omega_0^2 \int_0^t d\tau \int_0^\tau d\tau'\, \mathcal{L}_z(\tau')\,. \tag{1.13}$$

Further analysis will be possible after having found an equation for the LCF. But here some general remarks can be made about the expectations for the diffusion constant.

For the subsequent calculation, we suppose that the magnetic field is strong enough, so that the time $\tau_0 = 1/\Omega_0$ is the shortest characteristic time of the problem. Due to the fast oscillating multiple $e^{-i\,\overline{\Omega}_0\,\tau}$ the first term of the correlator (1.12a) does not give a significant contribution to the diffusion coefficient. A detailed analysis shows (see below) that the first term leads to the classical diffusion coefficient in an unperturbed magnetic field $D_{cl} = v_{th}^2\, \nu/\Omega_0^2$. An effect of magnetic field fluctuations is thus contained in the second term. The rate of decay of the correlation function is an outcome of two different mechanisms of decorrelation. They are collisions (due to the term $\nu\, t$ in the exponent) and field fluctuations (due to the function $\gamma(t)$ in the exponent). As one can see from the Eq. (1.6), this rate is crucial for the diffusion coefficient.

We also note the fact that the velocity correlator (1.12a) is almost the same as the one for guiding centers (see App. C, Eq. C.6), but, in contrast to the latter, it has an additional exponential multiple $e^{-\gamma(t)}$. This multiple leads to essential differences between the diffusion coefficient of particles and that of guiding centers.

From here we use Ω_0 instead of $\overline{\Omega}_0$ for brevity.

2 Lagrange correlation function of magnetic field

The right choice of a magnetic field correlation function $\mathcal{E}^{ij}(|\mathbf{r}_1 - \mathbf{r}_2|) = \langle \zeta^i(\mathbf{r}_1)\, \zeta^j(\mathbf{r}_2)\rangle$ is still an open question. Many authors who investigated neoclassical transport in magnetized plasma used

[1] Here we only emphasize again that the averaging procedure is made under the assumption of statistical independence of LCF and other stochastic processes. This assumption allows us to factorize the averaging of the velocity correlator in order to get an explicit expression in terms of LCF and MSD. This factorization procedure is analogous to the Corrsin approximation, and is reliable for a small coupling parameter ϵ.

the correlator in a Gaussian form. In the general case such a correlator can be presented as

$$\mathcal{E}^{ij}(\mathbf{r}) = \left\langle \zeta(\mathbf{r})^i\, \zeta(0)^j \right\rangle = C^{\,ij}(\mathbf{r})\, \exp\left(-\frac{x^2}{\lambda_\perp^2} - \frac{y^2}{\lambda_\perp^2} - \frac{z^2}{\lambda_\parallel^2} \right) ; \quad i = 1,\, 2,\, 3\,;$$

where the tensor function $C^{\,ij}(\mathbf{r})$ should be chosen to make the correlator $\mathcal{E}^{ij}(\delta\mathbf{r})$ satisfy the divergence-free condition. Detailed analysis of this problem can be found for example in [4], where a correlator for a two-dimensional field perturbation was proposed. In this paper it is shown that, although the Euler correlator $\mathcal{E}^{ij}(\delta\mathbf{r})$ has off-diagonal elements, the Lagrange correlation tensor (measured in a frame of reference moving with a magnetic field line) is a diagonal matrix.

One possible choice for the correlation matrix, satisfying the divergence-free condition, is

$$\mathcal{E}^{ij}(\mathbf{r}) = \left(\frac{\partial^2}{\partial r^i\, \partial r^j} - \delta^{ij}\, \frac{\partial^2}{\partial r^k\, \partial r^k} \right) C(\mathbf{r}) \tag{2.1}$$

with arbitrary function $C(\mathbf{r})$. It corresponds to the correlation function of the vector potential $\langle A(\mathbf{r})^i\, A(0)^j \rangle = C(\mathbf{r})\, \delta^{ij}$. Choosing $C(\mathbf{r}) = \exp(-x^2/\lambda_\perp^2 - y^2/\lambda_\perp^2 - z^2/\lambda_\parallel^2)$ and omitting "small" terms, we come to the following form of the ECF

$$\mathcal{E}^{ij}(\mathbf{r}) = \left\langle \zeta(\mathbf{r})^i\, \zeta(0)^j \right\rangle = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varsigma^2 \end{pmatrix} \exp\left(-\frac{x^2}{\lambda_\perp^2} - \frac{y^2}{\lambda_\perp^2} - \frac{z^2}{\lambda_\parallel^2} \right) ; \quad \mathbf{r} = \{x, y, z\}\,; \tag{2.2}$$

which we choose as a starting hypothesis for the subsequent calculations. Here $\varsigma^2 = \frac{2\,\lambda_\parallel^2}{\lambda_\perp^2 + \lambda_\parallel^2}$.

Notice that in the limit $\lambda_\perp \gg \lambda_\parallel$ the z-component of the magnetic field perturbation and the zz-component of ECF vanish. In this case we approach a degenerated 2-dimensional problem.

We use the Corrsin approximation [6, 7] to estimate the Lagrange correlator, given the Eulerian one:

$$\begin{aligned} \mathcal{L}(t)^{ij} = \left\langle \zeta(t)^i\, \zeta(0)^j \right\rangle &= \int \langle\, \zeta(\mathbf{r})^i\, \zeta(0)^j\, \delta(\mathbf{r} - \mathbf{r}(t)) \rangle\, d^3 r \\ &\approx \int \langle\, \zeta(\mathbf{r})^i\, \zeta(0)^j \rangle\, \langle \delta(\mathbf{r} - \mathbf{r}(t)) \rangle\, d^3 r\,. \end{aligned} \tag{2.3}$$

Here $\mathcal{L}(t)^{ij}$ is the Lagrange correlator of the magnetic field, $\langle \zeta(\mathbf{r})^i\, \zeta(0)^j \rangle$ – the Eulerian correlator, $\langle \delta(\mathbf{r} - \mathbf{r}(t)) \rangle$ is the averaged particle propagator. Treating $\mathbf{r}(t)$ as a stochastic variable, we apply the cumulant expansion to the Fourier representation of the δ-function to obtain

$$\langle \delta(\mathbf{r} - \mathbf{r}(t)) \rangle = \frac{\exp\left(-\frac{x^2}{2\,\langle x^2(t) \rangle} - \frac{y^2}{2\,\langle y^2(t) \rangle} - \frac{z^2}{2\,\langle z^2(t) \rangle} \right)}{(2\,\pi)^{3/2}\, \sqrt{\langle x^2(t) \rangle\, \langle y^2(t) \rangle\, \langle z^2(t) \rangle}}\,. \tag{2.4}$$

Here we used the assumption that the trajectory $\mathbf{r}(t)$ is a Gaussian process, i.e. the only nontrivial cumulants of the displacement $\mathbf{r}(t)$ are: $\langle\langle x^2(t) \rangle\rangle$, $\langle\langle y^2(t) \rangle\rangle$ and $\langle\langle z^2(t) \rangle\rangle$.

Now substituting (2.4) into (2.3) and performing the integration, we can express the diagonal elements of the Lagrange correlator in terms of the MSD:

$$\mathcal{L}_x(t) = \mathcal{L}_y(t) = \frac{1}{(1+\Gamma_x(t)/\lambda_\perp^2)\sqrt{\left(1+\Gamma_z(t)/\lambda_\parallel^2\right)}}\,, \tag{2.5}$$

$$\mathcal{L}_z(t) = \varsigma^2\,\mathcal{L}_x(t); \tag{2.6}$$

where we have designated $\Gamma_x(t) = \langle\!\langle x^2(t)\rangle\!\rangle$, $\Gamma_y(t) = \langle\!\langle y^2(t)\rangle\!\rangle$, $\Gamma_z(t) = \langle\!\langle z^2(t)\rangle\!\rangle$, and ς is defined after Eq. (2.2).

Next, taking the second time derivative of the Eq. (1.5), we obtain the equations for the MSD:

$$\Gamma''_\perp(t) = 2\,\mathbf{Re}\left[v_{th}^2\,e^{-\nu\,t-\gamma(t)}\,e^{-i\,\Omega_0\,t}\right] + 2\,\epsilon^2\,v_{th}^2\,\mathcal{L}_x(t)\,e^{-\nu\,t-\gamma(t)}\,, \tag{2.7}$$

$$\Gamma''_z(t) = 2\,v_{th}^2\,e^{-\nu\,t} + 4\,\epsilon^2\,v_{th}^2\,e^{-\nu\,t}\,\mathcal{L}_x(t)\,\cos\,\Omega_0 t\,, \tag{2.8}$$

where $\Gamma_\perp(t) = \Gamma_x(t) = \Gamma_y(t)$. These equations should be solved with initial conditions $\Gamma_i(0) = 0$, $\Gamma'_i(0) = 0$. The function $\gamma(t)$ is defined in Eq. (1.13).

The system of equations (2.5), (2.6), (2.7) and (2.8) completes the self consistent description of the MSD as well as LCF. One can solve this system with respect to the LCF and obtain an integral equation for the LCF, which stands on its own interest. An analogous procedure was performed by Hai-Da Wang *et al.* [4], where an integral equation was obtained for the LCF of a magnetic field line.

Another possibility is to exclude $\mathcal{L}(t)$ to obtain an equation for the MSD. This equation will lie in the focus of our interest in the next section.

3 Estimation of the MSD and diffusion coefficient

Let us first consider the motion in z-direction. From the equation (2.8) one can see that the second term in the right-hand-side is ϵ^2-small with respect to the first one and thus can be neglected. The equation becomes uncoupled and can be integrated, giving

$$\Gamma_z(t) = \frac{2\,v_{th}^2}{\nu^2}\,\left(\nu\,t + e^{-\nu\,t} - 1\right) = \begin{cases} v_{th}^2\,t^2\,, & \nu\,t \ll 1\,; \\ (2\,v_{th}^2/\nu)\,t\,, & \nu\,t \gg 1\,. \end{cases} \tag{3.1}$$

We see that the effect of the magnetic field fluctuations on the motion along the magnetic field appears as a small correction (of order less then ϵ^2) to the pure collisional result (3.1).

To analyze the perpendicular motion, we split equation (2.7) into two parts:

$$\Gamma_x(t) = \Gamma_{x1}(t) + \Gamma_{x2}(t)\,; \tag{3.2a}$$

$$\Gamma''_{x1}(t) = 2\,\mathbf{Re}\,v_{th}^2\,e^{-\nu\,\tau-\gamma(\tau)}\,e^{-i\,\Omega_0\,\tau}\,, \qquad \Gamma_{x1}(0) = 0\,,\ \Gamma'_{x1}(0) = 0\,; \tag{3.2b}$$

$$\Gamma''_{x2}(t) = 2\,\epsilon^2\,v_{th}^2\,\mathcal{L}_x(\tau)\,e^{-\nu\,\tau-\gamma(\tau)}\,, \qquad \Gamma_{x2}(0) = 0\,,\ \Gamma'_{x2}(0) = 0\,; \tag{3.2c}$$

3.1 Estimation of $\Gamma_{x1}(t)$.

Consideration of the first part $\Gamma_{x1}(t)$ is simplified due to the fast oscillating multiple $e^{-i\,\Omega_0\,t}$ in the right-hand-side of Eq. (3.2b). Performing a Fourier transform and noting that the spectrum of the function in the right-hand-side of the equation has a sharp maximum at $\omega = \Omega_0$, we expand the spectrum around this point. After inverse transform the equation can be integrated, giving

$$\Gamma_{x1}(t) = 2\,\rho_0^2\,\nu\,t + 2\,\rho_0^2\,\left(1 - \cos(\Omega_0\,t)\,e^{-\nu\,t-\gamma(t)}\right)\;, \tag{3.3}$$

where $\rho_0 = v_{th}/\Omega_0$ is the Lamor radius.

3.2 Estimation of $\Gamma_{x2}(t)$

To perform the analysis of $\Gamma_{x2}(t)$ it is useful to introduce the so-called *decorrelation time* t^*, that is defined as a solution of the equation $\nu\,t^* + \gamma(t^*) = 1$. For the rough estimation of the solution of the Eq. (3.2c) we replace $e^{-\nu\,\tau-\gamma(\tau)} \approx \theta(t^* - \tau)$. Thus for times $t < t^*$ the solution of the exact equation (3.2c) can be approximated by the solution of the subsidiary equation

$$\tilde{\Gamma}''_{x2}(t) = 2\,\epsilon^2\,v_{th}^2\,\mathcal{L}_x(t) = 2\,\epsilon^2\,v_{th}^2\left(1 + \frac{\Gamma_{x1}(t) + \tilde{\Gamma}_{x2}(t)}{\lambda_\perp^2}\right)^{-1}\left(1 + \frac{v_{th}^2}{\lambda_\parallel^2}\,t^2\right)^{-1/2}\;; \tag{3.4a}$$

$$\tilde{\Gamma}_{x2}(0) = 0\,,\ \tilde{\Gamma}'_{x2}(0) = 0\,; \tag{3.4b}$$

and for $t > t^*$ the solution is a linear function of time:

$$\Gamma_{x2}(t) \approx \begin{cases} \tilde{\Gamma}_{x2}(t)\,, & t \le t^*\,; \\ \tilde{\Gamma}_{x2}(t^*) + \tilde{\Gamma}'_{x2}(t^*)\,(t - t^*)\,, & t > t^*\,; \end{cases} \tag{3.5}$$

As soon as the solution of the equation (3.4) is found, the decorrelation time t^* can be obtained from the equation

$$\nu\,t^* + \frac{\varsigma^2}{2\,\rho_0^2}\,\tilde{\Gamma}_{x2}(t^*) = 1\,. \tag{3.6}$$

Indeed, the definition (1.13) of the function $\gamma(t)$ yields

$$\gamma(t) = \frac{\varsigma^2}{2\,\rho_0^2}\,\tilde{\Gamma}_{x2}(t)\,, \tag{3.7}$$

which gives rise to Eq. (3.6). The diffusion coefficient follows immediately from (3.3) and (3.5):

$$D_x = \frac{1}{2}\,\frac{d}{dt}\,\Gamma_x(t \to \infty) = \frac{1}{2}\,\Gamma'_{x1}(t \to \infty) + \frac{1}{2}\,\tilde{\Gamma}'_{x2}(t^*) = \frac{v_{th}^2\,\nu}{\Omega_0^2} + \frac{1}{2}\,\tilde{\Gamma}'_{x2}(t^*)\,. \tag{3.8}$$

Below we find some particular solutions of Eq. (3.4) and the corresponding diffusion coefficient in different limiting cases that are characterized by the ratio of the correlation lengths $\lambda_\parallel$ and $\lambda_\perp$.

3.2.1 The case $\lambda_\perp \to \infty$

As it was already mentioned, in this limiting case we have $\varsigma \to 0$, and the $\mathcal{L}_z(t)$ component of the LCF vanishes. As a result, the velocity decorrelation, caused by the magnetic field fluctuations, becomes very slow due to the multiplier ς appearing in the definition (1.13) of the function $\gamma(t)$.

Eq. (3.4a) can be rewritten in the form:

$$\tilde{\Gamma}''_{x2}(t) = 2\,\epsilon^2\, v_{th}^2 \left(1 + \Omega_b^2\, t^2\right)^{-1/2}, \tag{3.9}$$

giving rise to the solution

$$\tilde{\Gamma}_{x2}(t) = 2\,\epsilon^2\,\lambda_\parallel^2 \left(1 + \Omega_b\, t\, \mathrm{ArcSinh}(\Omega_b\, t) - \sqrt{1 + \Omega_b^2\, t^2}\right), \tag{3.10}$$

where we have introduced $\Omega_b = v_{th}/\lambda_\parallel\,,\;\; K_b = \epsilon\,\varsigma\,\Omega_0/\Omega_b$. The $\tilde{\Gamma}_{x2}(t)$ has the following asymptotic scaling:

$$\tilde{\Gamma}_{x2}(t) \approx \begin{cases} \epsilon^2\, v_{th}^2\, t^2\,, & \text{for} \quad t \ll 1/\Omega_b\,; \\ 2\,\epsilon^2\,\lambda_\parallel^2\,\Omega_b\, t\, \ln(\Omega_b\, t)\,, & \text{for} \quad t \gg 1/\Omega_b\,; \end{cases} \tag{3.11}$$

The dimensionless parameter K_b is the Kubo number, which characterizes the influence of the stochastic magnetic field on the averaged motion of a particle. Quantitatively it measures the contribution of magnetic fluctuations to the decorrelation of the particle velocity, and hence to the diffusion constant. To describe the influence of the collisions on the diffusion we introduce a dimensionless parameter $K_\nu = \nu/\Omega_b$. Considering different limiting cases of the parameters K_ν and K_b, we obtain the following transport regimes:

1. First we consider the case of weak perturbation $K_b^2 \ll K_\nu \ll 1$. The diffusion coefficient coincides up to a logarithmic correction factor with the well-known quasilinear result $D_{ql} \sim v_{th}\,\epsilon^2\,\lambda_\parallel$:

$$D_x \approx \frac{v_{th}^2\,\nu}{\Omega_0^2} + v_{th}\,\epsilon^2\,\lambda_\parallel\, \ln(\Omega_b/\nu)\,. \tag{3.12}$$

   The decorrelation time is $t^* \approx 1/\nu$.

2. In the case $K_\nu \ll K_b^2 \ll 1$ the transport regime is also of quasilinear type with a slightly different correction multiplier:

$$D_x \approx \frac{v_{th}^2\,\nu}{\Omega_0^2} + v_{th}\,\epsilon^2\,\lambda_\parallel\, \ln(\alpha)\,, \quad \text{where} \quad \alpha = \frac{1}{K_b^2\, \ln\left(1/K_b^2\right)}\,, \tag{3.13}$$

   and the decorrelation time becomes $t^* = \alpha/\Omega_b$.

3. The strong collisional case corresponds to the choice $K_\nu \gg \max\,(1,\, K_b)$. The corresponding decorrelation time is $t^* \approx 1/\nu$ and the diffusion constant

$$D_x \approx \frac{v_{th}^2\,\nu}{\Omega_0^2} + \frac{\epsilon^2\, v_{th}^2}{\nu}\,. \tag{3.14}$$

4. The last case $K_b \gg \max(1,\ K_\nu)$ considers strong magnetic fluctuations. The decorrelation time is $t^* \approx 1/(\epsilon\,\Omega_0\,\varsigma)$. In this case we observe a Bohm-type diffusion with characteristic scaling $D \sim B^{-1}$:

$$D_x \approx \frac{v_{th}^2\,\nu}{\Omega_0^2} + \frac{\epsilon\, v_{th}^2}{\varsigma\,\Omega_0}\,. \tag{3.15}$$

Now we can specify more carefully the condition $\lambda_\perp \to \infty$. To be able to neglect the terms in the first bracket in the right-hand-side of the Eq. (3.4a), the $\lambda_\perp$ should satisfy the condition:

$$\Gamma_x(t^*)/\lambda_\perp^2 \ll \min\left(v_{th}^2\,(t^*)^2/\lambda_\parallel^2\,,\ 1\right)\ , \tag{3.16}$$

where for rough estimation we can take $\Gamma_x(t^*) \approx D_x\, t^*$.

3.2.2 The case $\lambda_\parallel \to \infty$

Similarly to the previous case, the condition $\lambda_\parallel \to \infty$ can be written as

$$v_{th}^2\,(t^*)^2/\lambda_\parallel^2 \ll \min\left(\Gamma_x(t^*)/\lambda_\perp^2\,,\ 1\right), \tag{3.17}$$

where we substitute $\Gamma_x(t^*) \approx D_x\, t^*$, and the corresponding diffusion coefficients are calculated below.

The Eq. (3.4) in the considered limit becomes

$$\tilde{\Gamma}_{x2}''(t) = 2\,\epsilon^2\, v_{th}^2 \left(1 + \frac{\Gamma_{x1}(t) + \tilde{\Gamma}_{x2}(t)}{\lambda_\perp^2}\right)^{-1}\ ; \tag{3.18a}$$

$$\tilde{\Gamma}_{x2}(0) = 0\,,\ \tilde{\Gamma}_{x2}'(0) = 0\,; \tag{3.18b}$$

where the exact expression for $\Gamma_{x1}(t)$ is represented in the Eq. (3.3). Since we are only interested in the solutions of the equation above for times $t < t^*$, we can approximately substitute $\Gamma_{x1}(t) \approx 2\,\rho_0^2(1-\cos(\Omega_0\, t))$. The obtained equation has in its right-hand-side a slowly varying function of time with superimposed fast oscillations of small amplitude due to the term $(\rho_0^2/\lambda_\perp^2)\cos(\Omega_0\, t)$. So we can average the equation over the fast oscillations and obtain a simplified equation describing the slow dynamics of the $\tilde{\Gamma}_{x2}(t)$. Performing the integration over the period of oscillations we can treat $\tilde{\Gamma}_{x2}(t)$ as constant during the integration time. Making change of variables $U(t) = U_0 + \tilde{\Gamma}_{x2}(t)/(2\,\rho_0^2)$, we come to the following autonomous equation:

$$\begin{gathered} U''(t) = \frac{a^2}{2\,\sqrt{U^2(t)-1}}\,; \\ U'(0) = 0\,; \quad U(0) = U_0\,; \end{gathered} \tag{3.19}$$

where $a = \epsilon\, v_{th}\,\lambda_\perp/\rho_0^2,\ \ U_0 = 1 + \lambda_\perp^2/(2\,\rho_0^2)$. This equation can be integrated analytically:

$$\frac{\sqrt{\pi}}{2}\left(b\,\mathrm{Erfi}(W(t)) - \frac{1}{b}\,\mathrm{Erf}(W(t))\right) = a\,t\,; \tag{3.20}$$

where $b = U_0 + \sqrt{U_0^2 - 1}$, Erf($t$) and Erfi($t$) are error functions, and $\tilde{\Gamma}_{x2}(t)$ can be expressed through $W(t)$ as

$$\tilde{\Gamma}_{x2}(t) = \rho_0^2 \left(b\, e^{W^2(t)} + \frac{1}{b}\, e^{-W^2(t)} \right). \tag{3.21}$$

The solution for $\tilde{\Gamma}_{x2}(t)$, given by the Eqs. (3.20, 3.21), can now be used for the calculation of the decorrelation time t^* and the diffusion constant in different limiting cases. One readily finds an explicit expression for asymptotic behavior of $\tilde{\Gamma}_{x2}(t)$:

$$\tilde{\Gamma}_{x2}(t) = \begin{cases} \epsilon^2\, v_{th}^2\, \kappa^2\, t^2 & \Omega_\perp\, t \ll 1 \\ 2\,\epsilon\, v_{th}\, \lambda_\perp\, t\, \sqrt{\ln(\alpha\, \Omega_\perp t)} & \Omega_\perp\, t \gg 1 \end{cases}$$

where we have introduced the effective perpendicular magnetic frequency

$$\Omega_\perp = \frac{\epsilon\, v_{th}}{\lambda_\perp}\, \kappa^2\,, \qquad \kappa^2 = \frac{\lambda_\perp}{\sqrt{\lambda_\perp^2 + 4\,\rho_0^2}}\,, \qquad \alpha = \frac{8\,\kappa^2}{(1+\kappa^2)^2}.$$

Similarly to the previous section, we estimate the decorrelation time t^* from Eq. (3.6), where we substitute $\varsigma = \sqrt{2}$ in the limit considered, and find the diffusion constant from Eq. (3.8). There are four different asymptotic diffusion regimes, classified by the dimensionless parameters $K_{\perp\, b} = \epsilon\, \Omega_0\, \kappa / \Omega_\perp$ and $K_{\perp\, \nu} = \nu / \Omega_\perp$:

1. The case of weak perturbation $K_{\perp\, b}^2 \ll K_{\perp\, \nu} \ll 1$ leads to a Kadomtsev-Pogutse-type diffusion $D_{KP} \sim v_{th}\, \epsilon\, \lambda_\perp$ with a logarithmic correction. In this case we find $t^* \approx 1/\nu$, and the diffusion coefficient

$$D_x \approx \frac{v_{th}^2\, \nu}{\Omega_0^2} + v_{th}\, \epsilon\, \lambda_\perp\, \sqrt{\ln(\alpha\, \Omega_\perp / \nu)}\,. \tag{3.22}$$

2. In the case of $K_{\perp\, \nu} \ll K_{\perp\, b}^2 \ll 1$ the obtained result is also of Kadomtsev-Pogutse-type, but with different correction multiplier:

$$D_x \approx \frac{v_{th}^2\, \nu}{\Omega_0^2} + v_{th}\, \epsilon\, \lambda_\perp\, \sqrt{\ln(\beta)}\,, \quad \text{where} \quad \beta = \frac{\alpha}{K_{\perp\, b}^2\, \sqrt{\ln(\alpha / K_{\perp\, b}^2)}}\,. \tag{3.23}$$

The decorrelation time in this case is $t^* \approx 1/(K_{\perp\, b}^2\, \Omega_\perp\, \ln(\alpha / K_{\perp\, b}^2))$.

3. Strong collisional diffusion corresponds to the choice $K_{\perp\, \nu} \gg \max(1,\, K_{\perp\, b})$. In this case we find $t^* \approx 1/\nu$ and the diffusion coefficient:

$$D_x \approx \frac{v_{th}^2\, \nu}{\Omega_0^2} + \frac{\epsilon^2\, v_{th}^2\, \kappa^2}{\nu}\,. \tag{3.24}$$

4. The case $K_{\perp\, b} \gg \max(1,\, K_{\perp\, \nu})$ deals with strong magnetic fluctuations. We find a Bohm-type diffusion with characteristic scaling $D \sim B^{-1}$. The diffusion coefficient is

$$D_x \approx \frac{v_{th}^2\, \nu}{\Omega_0^2} + \frac{\epsilon\, v_{th}^2\, \kappa}{\Omega_0}\,, \tag{3.25}$$

and the decorrelation time t^* is given by $t^* \approx 1/(\epsilon\, \Omega_0\, \kappa)$.

4 Discussion

We have investigated some aspects of transport of charged particles in a stochastic magnetic field. On the base of the formal solution of a stochastic differential equation (A-Langevin equation) we obtained the velocity correlation function (1.12). It is expressed through the correlation functions of stochastic processes effecting the trajectory, such as stochastic magnetic field or random acceleration. Next we found an equation for the Lagrange correlation function of the magnetic field (Eq. (2.5)) in Corrsin approximation. This equation closes the system of equations for the determination of the mean square displacement as a function of time. The diffusion constant follows then as a time derivative of the MSD.

Since the equations obtained cannot be integrated in general case, we have considered only limiting cases with respect to the correlation lengths $\lambda_\perp$ and $\lambda_\parallel$. The results are presented in sections 3.2.1 and 3.2.2.

A structure of the diffusion constant becomes more clear from the following simple consideration. We see from Eq. (1.12) that the velocity correlator is a exponentially decreasing function of time with amplitude $\epsilon^2 v_{th}^2$. We conclude that the diffusion coefficient (1.6) can be roughly estimated as $D \sim \epsilon^2 v_{th}^2 t^*$, where t^* is the decorrelation time of the velocity correlator (the width of exponent). Hence it is the decorrelation time, what basically defines the transport rate.

The two mechanisms of the velocity decorrelation – magnetic field fluctuations and collisions – are characterized by the two dimensionless parameters (Kubo numbers): K_b and K_ν in the limiting case $\lambda_\perp \to \infty$, and $K_{\perp b}$ and $K_{\perp \nu}$ in the case $\lambda_\parallel \to \infty$. The transport regime (i.e., scaling of the diffusion coefficient) appears to be dependent solely on the ratio of these parameters.

We have classified the obtained results by the ratio of the Kubo numbers. In the case of "weak" perturbation, when both Kubo numbers are less then 1, we have found the well-known results: the quasilinear regime in the limit $\lambda_\perp \to \infty$, and the Kadomtsev-Pogutse result in the limit $\lambda_\parallel \to \infty$. In the case of "strong" perturbation the collisional or Bohm-like diffusion has been found. We note that the Rechester-Rosenbluth result is more complicated to obtain analytically, since it cannot be calculated in the limiting cases $\lambda_\perp \to \infty$ or $\lambda_\parallel \to \infty$.

It is interesting to note that the limiting case $K_b \gg \max(1,\, K_\nu)$ includes also a theoretical limit $\Omega_0 \to \infty$. In this case we find the diffusion constant vanish, yet the guiding center diffusion does not (see the note at the end of the App. C). Indeed, the stronger the magnetic field is, the better the approximation holds true that a single particle describes a spiral path along a magnetic field line, i.e. the plasma is frozen in a strong magnetic field. As long as we keep the perturbation parameter ϵ constant, the field lines keep diffusing, and so do the guiding centers of particles. The spurious contradiction is due to the well-known phenomenon, that in a magnetized plasma the transport of guiding centers differs from the transport of particles themselves [8]. We see that there exists only a region of plasma parameters, beyond which the guiding center approximation fails to describe the transport processes in magnetized plasma.

A Derivation of the velocity correlation function

The velocity correlation function (VCF) can be obtained from the exact solution (1.9). For this purpose we construct a product of two components of the velocity $u(t_1)^i\, u(t_2)^j$ and then average over the stochastic processes. As it was mentioned in the main text (see the footnote on the p. 4), it is not possible to perform the averaging over different stochastic processes independently, since the magnetic field $\vec{\zeta}(\mathbf{r}(t))$, appearing in the solution (1.9), is an implicit function of all other stochastic values. However, if the magnetic fluctuations are small, we can make the hypothesis that the Lagrange correlator of the magnetic field is statistically independent of the other stochastic values – initial velocity $\mathbf{u}_0$ and random acceleration $\mathbf{a}(t)$. This hypothesis allows the factorization of the velocity correlation function, somewhat analogous to the well-known Corrsin approximation.

First we average the product $u(t_1)^i\, u(t_2)^j$ over the initial velocities $\mathbf{u}_0$ and random accelerations $\mathbf{a}(t)$. For the averaging, we use the following assumption about initial velocity distribution:

$$\langle u_0^i \rangle = 0; \quad \langle u_0^i\, u_0^j \rangle = v_{th}^2\, \delta^{ij}\,. \tag{A.1}$$

A calculation leads to the following expression for the correlation matrix in index notation:

$$\left\langle u(t_1)^i\, u(t_2)^j \right\rangle_{u_0,\, a(t)} = v_{th}^2\; e^{-\nu\,(t_1 - t_2)}\, R_3\Big(-\Omega_0\, t_1\Big)^i_{\ k}\, G(t_2, t_1)^k_{\ l}\, R_3\Big(\Omega_0\, t_2\Big)^l_{\ j}\,, \tag{A.2}$$

where $R_3(t) = \exp(t\, L_3)$ is a finite rotation matrix, and the propagator $G(t_2, t_1)$ is defined in the Eq. (1.10). To obtain this result, the following properties of the matrix $G(t_2, t_1)$ were used:

$$G(t)^T = G(t)^{-1}\,, \qquad G(t', t_1)\, G(t', t_2)^T = G(t_2, t_1)\,.$$

Both equations follow from the general properties of a time-ordered exponent and from the fact that $G(t_2, t_1)$ is an exponent of a skew symmetric matrix.

Next the equation (A.2) should be averaged over the stochastic magnetic field $\vec{\zeta}(\mathbf{r}(t), t)$, which appears in the exponent of the propagator $G(t_2, t_1)$. To calculate the average of the exponent of a stochastic variable, we need to apply the cumulant expansion. The generalization of the cumulant expansion [1] on the time-ordered exponent is

$$\begin{aligned} &\left\langle T\, \exp\left(\int_0^t V(\tau)\, d\tau\right)\right\rangle = \\ &\quad \exp\left(\int_0^t dt_1\, \langle V(t_1)\rangle + \frac{1}{2!}\iint_0^t dt_1\, dt_2\, T\langle\!\langle V(t_1)\, V(t_2)\rangle\!\rangle + \cdots\right) = \\ &\quad \exp\left(\int_0^t dt_1\, \langle V(t_1)\rangle + \int_0^t dt_1 \int_0^{t_1} dt_2\, \langle\!\langle V(t_1)\, V(t_2)\rangle\!\rangle + \cdots\right), \end{aligned} \tag{A.3}$$

where $T\langle\!\langle\, \cdot\, \rangle\!\rangle := \langle\!\langle T\, \cdot\, \rangle\!\rangle$ is a time-ordered cumulant. Now we apply this formula to the Eq. (A.2), substituting $V(t) = -\epsilon\, \Omega_0\, \hat{H}_{int}(t)$. Making use of the definitions (1.4), (1.7) and (1.11), we obtain

$$\langle G(t_2, t_1)\rangle = \langle G(\tau)\rangle = \exp\left\{-\epsilon^2\, \Omega_0^2 \int_0^\tau (\tau - \tau')\, A(\tau')\, d\tau'\right\}\,, \tag{A.4}$$

where

$$A(\tau') = \begin{pmatrix} \mathcal{L}_x(\tau')\,\cos(\Omega_0\,\tau') + \mathcal{L}_z(\tau') & -\mathcal{L}_x(\tau')\,\sin(\Omega_0\,\tau') & 0 \\ \mathcal{L}_x(\tau')\,\sin(\Omega_0\,\tau') & \mathcal{L}_x(\tau')\,\cos(\Omega_0\,\tau') + \mathcal{L}_z(\tau') & 0 \\ 0 & 0 & 2\,\mathcal{L}_x(\tau')\,\cos(\Omega_0\,\tau') \end{pmatrix},$$

and $\tau = |t_1 - t_2|$. The matrix in the exponent has a reduced form, and hence we have two invariant subspaces: $\{x, y\}$ and $\{z\}$. In complex number representation (see App. B) we replace in the $\{x, y\}$ aggregate:

$$\hat{L}_3 \to i\,; \qquad (\hat{L}_3)^2 \to -1\,; \qquad \hat{R}_3(\alpha) \to e^{i\,\alpha}\,;$$

Thus, the propagator can now be rewritten in the form:

$$\langle G(\tau)\rangle = \tag{A.5}$$
$$\exp\left\{-\epsilon^2\,\Omega_0^2\int_0^\tau(\tau-\tau')\begin{pmatrix} \left(\mathcal{L}_x(\tau')\,e^{i\,\Omega_0\,\tau'} + \mathcal{L}_z(\tau')\right) & & 0 \\ & & 0 \\ 0 & 0 & 2\,\mathcal{L}_x(\tau')\,\cos(\Omega_0\,\tau') \end{pmatrix} d\tau'\right\}.$$

The diagonal elements of the $\{x, y\}$ aggregate of the propagator can be obtained as the real part of the complex valued expression above.

Note that, after averaging, the matrix R_3 commutes with $\langle G(\tau)\rangle$, and thus the whole velocity correlator becomes:

$$\langle u(t_1)^i\,u(t_2)^j\rangle = \left\langle u(\tau)^i\,u(0)^j\right\rangle = v_{th}^2\,e^{-\nu\,\tau}\,R_3(-\Omega_0\,\tau)\;\langle G(\tau)\rangle\;; \quad \tau = |t_1 - t_2|\,; \tag{A.6}$$

Now we can write the results of integration for $\{x, y\}$ and $\{z\}$ subspaces independently. For diagonal elements we get:

$$\langle u(t)\,u(0)\,\rangle = \begin{cases} \mathbf{Re}\left[v_{th}^2\,e^{-\nu\,t}\,e^{-i\,\Omega_0\,t}\,e^{-\gamma(t)}\,e^{-\mu(t)}\right]; & \text{for} \quad \{x, y\} \\ v_{th}^2\,e^{-\nu\,t}\,e^{-2\,\mathbf{Re}[\mu(t)]}; & \text{for} \quad \{z\}\,. \end{cases} \tag{A.7}$$

Here we have used the designations:

$$\gamma(t) = \epsilon^2\,\Omega_0^2\int_0^t(t-\tau)\mathcal{L}_z(\tau)\,d\tau = \epsilon^2\,\Omega_0^2\int_0^t d\tau\int_0^\tau d\tau'\,\mathcal{L}_z(\tau')\,; \tag{A.8}$$

$$\mu(t) = \epsilon^2\,\Omega_0^2\int_0^t(t-\tau)\mathcal{L}_x(\tau)\,e^{i\,\Omega_0\,\tau}\,d\tau = \epsilon^2\,\Omega_0^2\int_0^t d\tau\int_0^\tau d\tau'\,\mathcal{L}_x(\tau')\,e^{i\,\Omega_0\,\tau'}\,. \tag{A.9}$$

The formula for the velocity correlator can be simplified if we use the reasonable assumption that the Lamor frequency Ω_0 defines the shortest characteristic time in the system (limit of strong magnetic field). Noting that the Fourier spectra of the integrated function has a sharp maximum at $\omega = \Omega_0$, we come to the asymptotic (as $\Omega_0 \to \infty$) formula for $\mu(t)$:

$$\mu(\tau) \;\;=\;\; \epsilon^2\,i\,\Omega_0\,\tau + \epsilon^2\left(1 - \mathcal{L}_x(\tau)\,e^{i\,\Omega_0\,\tau}\right). \tag{A.10}$$

The first term in this expression has, in fact, the form of a secular term of a perturbation expansion. It can be nullified by the renormalization of the frequency Ω_0. Designating $\overline{\Omega}_0 = \Omega_0 + \epsilon^2 \Omega_0$ and omitting the non-relevant terms of the order ϵ^2, we can write for the $\{x, y\}$ component of the correlator:

$$\langle u(t_1)\, u(t_2) \rangle = \mathbf{Re}\, v_{th}^2\, e^{-\nu\tau - \gamma(\tau)}\, e^{-i\overline{\Omega}_0\tau} + \epsilon^2\, v_{th}^2\, \mathcal{L}_x(\tau)\, e^{-\nu\tau-\gamma(\tau)}\,. \tag{A.11}$$

In an analogous way we obtain an expression for the $\{z\}$ component. Without loss of accuracy we can replace Ω_0 by $\overline{\Omega}_0$ and omit non-relevant corrections of the order ϵ^2, finally coming to

$$\langle u(t_1)\, u(t_2) \rangle = v_{th}^2\, e^{-\nu\tau} + 2\,\epsilon^2\, v_{th}^2\, e^{-\nu\tau}\, \mathcal{L}_x(\tau)\, \cos\, \overline{\Omega}_0\tau\,. \tag{A.12}$$

This result is similar to the one, obtained for guiding centers correlation function (see App. C).

B Complex number representation

Making use of the homomorphism φ:

$$\varphi : A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \rightarrow z = a + ib$$

we can replace for the calculation in the $\{x, y\}$ plane:

$$\hat{L}_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rightarrow i\,; \qquad R_3(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \rightarrow e^{i\,\alpha}\,.$$

By the inverse transformation the diagonal elements of the matrix are just a real part of the complex number:

$$z \rightarrow A = \begin{pmatrix} \mathbf{Re}[z] & -\mathbf{Im}[z] \\ \mathbf{Im}[z] & \mathbf{Re}[z] \end{pmatrix}$$

C Guiding Center Diffusion

In this section we calculate the velocity correlation function $\langle u(t_1)^i\, u(t_2)^j \rangle$ in a guiding center approximation for particles, moving in a slightly perturbed homogeneous magnetic field. In the limit of a strong magnetic field, a single particle performs a spiral motion strictly along a magnetic field line. If the magnetic field is unperturbed ($\mathbf{B}_0 = B_0\, \mathbf{e}_z$), the velocity correlator has a form:

$$\langle u_0(t)^i\, u_0(0)^j \rangle = \langle \mathbf{u}_0(t) \otimes \mathbf{u}_0(0) \rangle = v_{th}^2\, e^{-\nu\, t}\, e^{-\Omega_0\, t\, \hat{L}_3}\,;$$

where $e^{-\Omega_0\, t\, L_3} = R_3(-\Omega_0\, t)$ is a finite rotation matrix and Ω_0 is a Lamor frequency. The subscript "0" is used for the unperturbed velocities.

Next we perturb the magnetic field, adding a small stochastic component:

$$\mathbf{B}(t) = B_0\,\mathbf{e}_z + \epsilon\,B_0\,\vec{\zeta}(t)\,.$$

We suppose that the perturbation $\vec{\zeta}(t)$ yields the following Lagrange correlation function:

$$\langle\vec{\zeta}(t)^i\,\vec{\zeta}(0)^j\rangle = \begin{pmatrix} \mathcal{L}_x(t) & & \\ & \mathcal{L}_x(t) & \\ & & \mathcal{L}_z(t) \end{pmatrix}. \tag{C.1}$$

Since the perturbation of the magnetic field is small, it can be considered as a rotation of the unperturbed vector field $\mathbf{B}_0$ at each point of space (or time on Lagrange trajectory) to some angle α without change of its amplitude. This rotation can be represented as

$$\mathbf{B}(t) = e^{\mathbf{k}(t)\cdot\hat{\mathbf{L}}}\,\mathbf{B}_0$$

where $\mathbf{k}$ is the rotation vector: $|\mathbf{k}| = \alpha$ is the rotation angle, and $\mathbf{n} = \mathbf{k}/k$ is the rotation axis. It is straightforward to calculate (for $\epsilon \longrightarrow 0$):

$$\mathbf{k}(t) = \epsilon\,[\vec{\zeta}(t)\times\mathbf{e}_z]\,. \tag{C.2}$$

Using the fact that in the strong magnetic field the guiding center follows exactly the magnetic field line, we need not to calculate the perturbed trajectory of the particle, rather we just have to apply the same rotation operator $e^{\mathbf{k}(t)\cdot\hat{\mathbf{L}}}$ to the unperturbed velocities. The "perturbed" correlator becomes:

$$\langle u(t)^i\,u(0)^j\rangle = \langle e^{\mathbf{k}(t)\cdot\hat{\mathbf{L}}}\,\mathbf{u}_0(t)\otimes e^{\mathbf{k}(0)\cdot\hat{\mathbf{L}}}\,\mathbf{u}_0(0)\rangle \tag{C.3}$$

Next we expand the rotation matrices, using the fact that $|\mathbf{k}| \ll 1$:

$$e^{\mathbf{k}(t)\cdot\hat{\mathbf{L}}} \approx 1 + \mathbf{k}(t)\cdot\hat{\mathbf{L}}\,.$$

Substituting this expansion in the expression for the velocity correlator, we obtain in index notation:

$$u(t)^i\,u(0)^j = u_0(t)^i\,u_0(0)^j + [...] + k(t)^\alpha\,k(0)^\beta\,L_{\alpha\,l}^{\;\,i}\,L_{\beta\,m}^{\;\,j}\,u_0(t)^l\,u_0(0)^m\,, \tag{C.4}$$

where $[...]$ implies terms proportional to the first order of $\mathbf{k}(t)$.

The vector $\mathbf{k}(t)$ is a stochastic function, which correlators can be readily found from Eqs. (C.1) and (C.2):

$$\langle\mathbf{k}(t)\rangle \equiv 0;\quad \langle k(t)^i\,k(0)^j\rangle = \epsilon^2\,\mathcal{L}_x(t)\left(\delta^{ij} - n_z^i\,n_z^j\right), \tag{C.5}$$

where $n_z = \{0,0,1\}$. Now we can average the expression (C.4) over magnetic fluctuations. Again we have to use the Corrsin approximation, which justifies the independent averaging over

stochastic field (here over $\mathbf{k}(t)$) and over the trajectory variables (here over the unperturbed velocities $\mathbf{u}_0(t)$). In this connection $\langle u_0(t)^i\, u_0(0)^j\rangle$ is the correlator on unperturbed trajectory. Then, after straightforward but cumbersome calculations, we get:

$$\langle u(t)^i\, u(0)^j\rangle = v_{th}^2\, e^{-\nu\, t} \left(R_3(-\Omega_0\, t) + 2\,\epsilon^2\,\mathcal{L}_x(t)\, \cos(\Omega_0\, t) + \epsilon^2\,\mathcal{L}_x(t) \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} \right). \quad \text{(C.6)}$$

Comparison of this expression with Eqs. (A.11) and (A.12) shows that the z-component of the guiding center correlator coincides exactly with that of particles. Yet the $\{x, y\}$ component of the particle correlator has an additional multiple $e^{-\gamma(t)}$. In contrast to the particle diffusion, the last term in the Eq. (C.6) gives a non-vanishing in the limit $\Omega_0 \to \infty$ contribution to the guiding center diffusion coefficient $D_x = \int_0^\infty \langle u^x(\tau)\, u^x(0)\rangle\, d\tau$.

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